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# Common stabilizations of Heegaard splittings of link exteriors <sup>☆</sup>

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## Abstract

We give a condition for a pair of unknotting tunnels of a non-trivial tunnel number one link to give a genus three Heegaard splitting of the link complement and show that every 2-bridge link has such a pair of unknotting tunnels.

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## 1. Introduction

A *compression body*  $H$  is a 3-manifold obtained from a compact connected closed surface  $S$  by attaching 2-handles to  $S \times I$  on  $S \times \{1\}$  and capping off any resulting 2-sphere boundary components with 3-handles.  $S \times \{0\}$  is denoted by  $\partial_+ H$  and  $\partial H - \partial_+ H$  is denoted by  $\partial_- H$ . A compression body  $H$  is called a *handlebody* if  $\partial_- H = \emptyset$ .

If a compact 3-manifold  $M$  is the union of two compression bodies  $H_1$  and  $H_2$  along their common “plus” boundary  $S = \partial_+ H_1 = \partial_+ H_2$ , we call the decomposition  $M = H_1 \cup_S$

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$H_2$  a Heegaard splitting of  $M$  and  $S$  a Heegaard surface of  $M$ . The minimum number of the genus of  $S$  among all Heegaard splittings of  $M$  is called the *Heegaard genus* (or *genus*) of  $M$ .

Suppose  $H_1 \cup_S H_2$  is a Heegaard splitting of a 3-manifold  $M$  and  $\alpha$  is a properly embedded arc in  $H_2$  parallel to an arc in  $S$ . That is, there is an embedded disk  $D$  in  $H_2$  whose boundary is the union of  $\alpha$  and an arc in  $\partial_+ H_2$ . Now add a neighborhood of  $\alpha$  to  $H_1$  and delete it from  $H_2$ . Once again the result is a Heegaard splitting  $H'_1 \cup_{S'} H'_2$ , where the genus of each  $H'_i$  is one greater than  $H_i$ . This process is called a *stabilization* of  $S$ .

Every compact 3-manifold can be triangulated and any two triangulations of a 3-manifold are PL-equivalent [1,7]. It follows that every compact 3-manifold has a Heegaard splitting and any two Heegaard splittings of a 3-manifold have a common stabilization. In fact, there is no example of distinct Heegaard splittings of a same closed 3-manifold which cannot be made isotopic by a single stabilization of one of the splittings, and sufficient stabilizations of the other to ensure that the genus of the two surfaces is the same. This makes the following conjecture very optimistic.

**Conjecture 1.1** [9]. Suppose  $H_1 \cup_S H_2$  and  $H'_1 \cup_{S'} H'_2$  are Heegaard splittings of the same 3-manifold of, genus  $g \leq g'$  respectively. Then the splittings obtained by one stabilization of  $S'$  and  $g' - g + 1$  stabilizations of  $S$  are isotopic.

A *tunnel system* (or *tunnels*) of a knot or a link  $K$  is a collection of disjoint embedded arcs  $t_1, t_2, \dots, t_n$  in  $S^3$  with  $K \cap \bigcup_{i=1}^n t_i = \bigcup_{i=1}^n \partial t_i$  such that  $H = S^3 - N(K \cup \bigcup_{i=1}^n t_i)$  is a genus  $n + 1$  handlebody. (Here  $N(X)$  denotes a regular neighborhood of  $X$ .) The tunnel system gives rise to a Heegaard splitting of the exterior of  $K$

$$\overline{S^3 - N(K)} = H \cup_{\partial H} \overline{N\left(K \cup \bigcup_{i=1}^n t_i\right) - N(K)}$$

where  $N(K)$  is contained in the interior of  $N(K \cup \bigcup_{i=1}^n t_i)$ . The minimum of such number  $n$  is called the *tunnel number* of  $K$ . If the tunnel number of  $K$  is 1, the tunnel is called an *unknotting tunnel* of  $K$ .

For a tunnel number one knot  $K$ , we consider two non-isotopic unknotting tunnels  $t_1, t_2$  and corresponding Heegaard surfaces  $S_1, S_2$  of the exterior of  $K$ . Now suppose  $H = S^3 - N(K \cup t_1 \cup t_2)$  is a genus three handlebody. This means that  $S' = \partial H$  becomes a Heegaard surface for the genus two handlebodies  $\overline{S^3 - N(K \cup t_1)}$  and  $\overline{S^3 - N(K \cup t_2)}$ . By [10], there is at most one Heegaard splitting of a handlebody of a given genus. This implies  $S'$  is a common stabilization of  $S_1$  and  $S_2$  and shows a validity of Conjecture 1.1. There are examples of knots having this property—torus knots and 2-bridge knots.

In this paper, we give a sufficient condition for tunnel number one links to have this property and show that 2-bridge links satisfy this condition.

## 2. Examples

### 2.1. Torus knots

A *torus knot* is a knot on the standard torus embedded in  $S^3$ . A torus knot can be characterized by two relatively prime integers  $p$  and  $q$ .  $K_{p,q}$  is a torus knot that winds the standard torus  $p$  times in meridional direction and  $q$  times in longitudinal direction. A torus knot has 3 types of unknotting tunnels  $t_p, t_q, t_0$  (Fig. 1) and they are classified in [2].

**Theorem 2.1** [2] (Boileau–Rost–Zieschang). *Let  $K_{p,q}$  be a torus knot of type  $(p, q)$ , where  $\gcd(p, q) = 1$  and  $p > q > 1$ .*

- (1) Any unknotting tunnel of  $K_{p,q}$  is isotopic to  $t_p, t_q$  or  $t_0$ .
- (2)  $t_0$  is isotopic to  $t_p \iff q \equiv \pm 1 \pmod{p}$ .
- (3)  $t_0$  is isotopic to  $t_q \iff p \equiv \pm 1 \pmod{q}$ .
- (4)  $t_p$  is isotopic to  $t_q \iff |p - q| = 1$ .

In [2], they also proved that the two Heegaard splittings given by any two unknotting tunnels among  $t_p, t_q, t_0$  of a torus knot, say  $t_p$  and  $t_q$ , have a common stabilization  $S' = \partial N(K \cup t_p \cup t_q)$ .

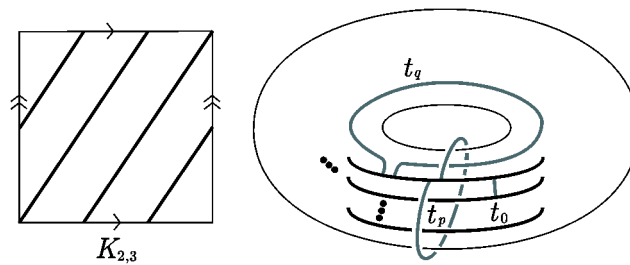


Fig. 1.  $K_{2,3}$  and unknotting tunnels of a torus knot.

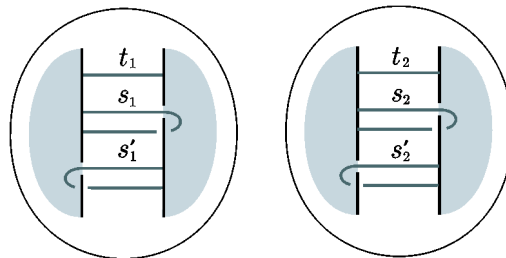


Fig. 2. Unknotting tunnels of a 2-bridge knot.

## 2.2. 2-bridge knots

$S^3$  can be understood as a gluing of two 3-balls along the boundary spheres. A *2-bridge knot* is a knot which can be decomposed into two trivial 2-string tangles in those two 3-balls. A 2-bridge knot has 6 types of unknotting tunnels  $s_1, s'_1, t_1, s_2, s'_2, t_2$  (Fig. 2) and they are classified in [4,8].

In [3], Hagiwara showed that the two Heegaard splittings given by any two unknotting tunnels among  $t_1, s_1, s'_1, t_2, s_2, s'_2$  of a 2-bridge knot have a common stabilization. Also in [5], it is shown by Kobayashi that all splittings of 2-bridge knot exteriors of genus greater than 2 are reducible.

## 3. Common stabilizations of tunnel number one link exterior

When we have two disjoint unknotting tunnels  $t_1, t_2$  of a knot  $K$ ,  $\partial N(K \cup t_1 \cup t_2)$  may not be a Heegaard surface even if  $t_1$  and  $t_2$  are isotopic tunnels. Take  $t_2$  as a parallel copy of  $t_1$ . Pull a part of  $t_2$  in a complicated way and hook it to  $t_1$ . This construction does not give a genus three Heegaard surface (Fig. 3). So there must be some restrictions on the choice of the unknotting tunnels.

A tunnel number one link has two components. If both components are unknotted, it is a 2-bridge link [6]. There are infinitely many tunnel number one links with one knotted component or two knotted components. For example, we can make one with two knotted components from a non-trivial tunnel number one knot by sliding an endpoint of the tunnel over the knot in longitudinal direction many times (Fig. 4).

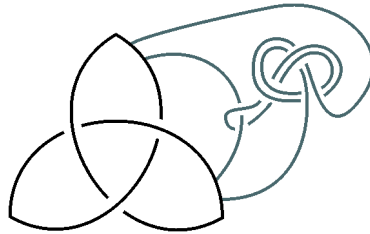


Fig. 3. Embedding of tunnels which does not give a genus three splitting.

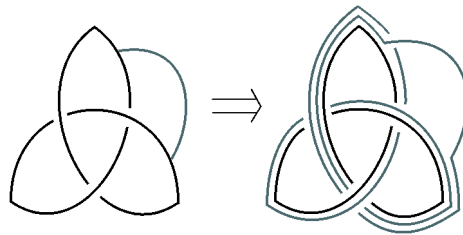


Fig. 4. Making a tunnel number one link with two knotted components.

Here we give a sufficient condition for a non-trivial tunnel number one link exterior to have a common stabilization when we have two disjoint unknotting tunnels. Some arguments in the proof work only for 2-component links.

**Theorem 3.1.** *Let  $L = K_1 \cup K_2$  be a non-trivial tunnel number one link and  $t_1$  and  $t_2$  be two disjoint unknotting tunnels of  $L$  such that a meridian disk  $D$  of the genus two handlebody  $H_1 = S^3 - N(L \cup t_1)$  does not intersect  $t_2$ . Then  $S^3 - N(L \cup t_1 \cup t_2)$  is a genus three handlebody.*

#### 4. Proof of Theorem 3.1

Let  $H_2$  be the genus two handlebody  $S^3 - N(L \cup t_2)$ .  $\partial H_1$  (respectively  $\partial H_2$ ) consists of three parts as in Fig. 5—two once-punctured tori  $T_{1,K_1}$  and  $T_{1,K_2}$  (respectively  $T_{2,K_1}$  and  $T_{2,K_2}$ ) and an annulus  $A_1$  (respectively  $A_2$ ) joining them. Let  $E_1$  and  $E_2$  be two non-separating meridian disks of  $H_2$  which are not parallel to each other. Then  $E_1 \cup E_2$  cuts  $H_2$  into a 3-ball. We may choose such  $E_1$  and  $E_2$  so as to satisfy  $\partial T_{1,K_1} \cap E_i = \emptyset$  ( $i = 1, 2$ ) and  $\partial T_{1,K_2} \cap E_i = \emptyset$  ( $i = 1, 2$ ) and minimize  $|D \cap (E_1 \cup E_2)|$  which is the number of components of  $D \cap (E_1 \cup E_2)$ .

**Lemma 4.1.**  *$\partial D \cap A_1$  and  $\partial E_i \cap A_2$  ( $i = 1, 2$ ) are non-empty. We may assume that  $\partial D \cap T_{1,K_1}$ ,  $\partial D \cap T_{1,K_2}$ ,  $\partial D \cap A_1$  consist of essential arcs and also  $\partial E_i \cap T_{2,K_1}$ ,  $\partial E_i \cap T_{2,K_2}$ ,  $\partial E_i \cap A_2$  ( $i = 1, 2$ ) consist of essential arcs.*

**Proof.** Suppose  $\partial D \cap A_1 = \emptyset$ . Then  $\partial D$  is in  $\partial N(K_1)$  or  $\partial N(K_2)$ , say  $\partial D \subset \partial N(K_1)$ . If  $\partial D$  is a meridian of  $\partial N(K_1)$ , a punctured  $S^2 \times S^1$  is in  $S^3$ , a contradiction. If  $\partial D$  is a longitude of  $\partial N(K_1)$ , then the regular neighborhood of  $N(K_1) \cup D$  is a 3-ball containing  $K_1$  only. This implies  $S^3 - N(L)$  is reducible, so  $L$  is a non-trivial split link, hence cannot have an unknotting tunnel, a contradiction. If  $\partial D$  winds around  $\partial N(K_1)$  longitudinally more than once, then a punctured lens space is in  $S^3$ , a contradiction. This proves  $\partial D \cap A_1 \neq \emptyset$ . Similarly  $\partial E_i \cap A_2 \neq \emptyset$  ( $i = 1, 2$ ).

If  $\partial D$  meets  $A_1$  in an essential loop, let  $D'$  be the meridian disk of  $t_1$  that the essential loop bounds. Then the 2-sphere  $D \cup D'$  separates  $L$ , so  $L$  is a non-trivial split link, hence

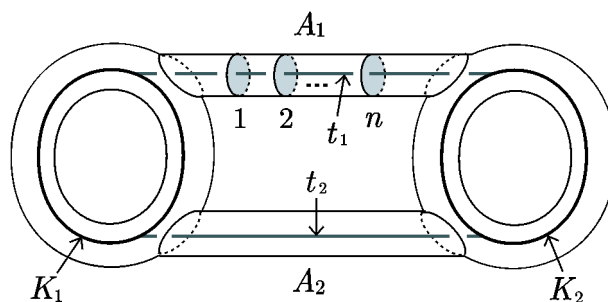


Fig. 5.  $N(L \cup t_1) \cap \text{int}(E_1 \cup E_2)$ .

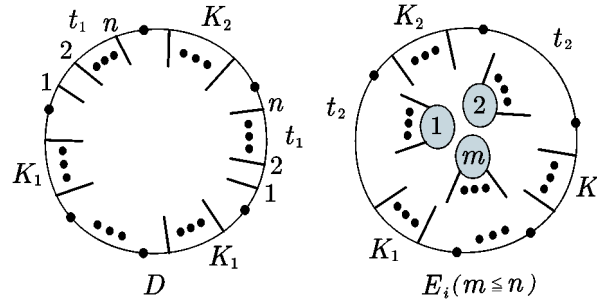


Fig. 6.  $D \cap (E_1 \cup E_2)$  and  $D \cap E_i$  ( $i = 1, 2$ ) with labels.

cannot have an unknotting tunnel, a contradiction. Therefore  $\partial D \cap A_1$  has no essential loops. Similarly  $\partial E_i \cap A_2$  ( $i = 1, 2$ ) has no essential loops.

If any of  $\partial D \cap T_{1,K_1}$ ,  $\partial D \cap T_{1,K_2}$ ,  $\partial D \cap A_1$ ,  $\partial E_i \cap T_{2,K_1}$ ,  $\partial E_i \cap T_{2,K_2}$ ,  $\partial E_i \cap A_2$  ( $i = 1, 2$ ) have inessential arcs, we can remove them by isotopy. So we may assume that all the intersections are essential arcs.  $\square$

Label the arcs  $\partial D \cap T_{1,K_1}$ ,  $\partial D \cap T_{1,K_2}$ ,  $\partial D \cap A_1$  of  $\partial D$  with  $K_1$ ,  $K_2$ ,  $t_1$ , respectively. Also label the arcs  $\partial E_i \cap T_{2,K_1}$ ,  $\partial E_i \cap T_{2,K_2}$ ,  $\partial E_i \cap A_2$  of  $E_i$  ( $i = 1, 2$ ) with  $K_1$ ,  $K_2$ ,  $t_2$ , respectively. Let us assume that  $t_1$  intersects  $E_1 \cup E_2$  transversely in  $n$  points (possibly  $n$  can be zero) and number the meridian disks  $N(L \cup t_1) \cap \text{int}(E_1 \cup E_2)$  of  $t_1$  consecutively along  $N(L \cup t_1)$  (Fig. 5).

Since  $D$  does not intersect  $t_2$  by hypothesis,  $D \cap (E_1 \cup E_2)$  consists of loops and properly embedded arcs in  $D$ . If there is a circle component of  $D \cap (E_1 \cup E_2)$  in  $D$ , by cutting and pasting  $E_1 \cup E_2$  along an innermost disk in  $D$ , we can reduce  $|D \cap (E_1 \cup E_2)|$ .

So we may assume that  $D \cap (E_1 \cup E_2)$  consists of properly embedded arcs in  $D$ . An endpoint of an arc of  $D \cap (E_1 \cup E_2)$  which is in an arc of  $\partial D$  labelled  $t_1$  corresponds to a meridian disk  $N(L \cup t_1) \cap \text{int}(E_1 \cup E_2)$  of  $t_1$ . Label that endpoint with the number given to the corresponding meridian disk. Fig. 6 shows the intersection  $D \cap (E_1 \cup E_2)$  and  $D \cap E_i$  ( $i = 1, 2$ ) on  $D$  and  $E_i$  ( $i = 1, 2$ ), respectively.

**Lemma 4.2.** Any arc of  $\partial D$  labelled  $K_1$  or  $K_2$  has intersection with  $E_1 \cup E_2$ .

**Proof.** Suppose an arc  $a$  labelled  $K_1$  has no intersection with  $E_1 \cup E_2$ . Let  $b$  be the subarc of  $\partial T_{1,K_1}$  joining two endpoints of  $\partial a$  (Fig. 7). Here we use the fact that  $L$  is a 2-component link—an arc labelled  $K_1$  (respectively  $K_2$ ) together with a subarc of  $\partial T_{1,K_1}$  (respectively  $\partial T_{1,K_2}$ ) makes a closed curve.

Cut the genus 2-handlebody  $H_2 = \overline{S^3 - N(L \cup t_2)}$  by  $E_1 \cup E_2$ . Then we get a 3-ball. Since  $a \cup b$  does not intersect  $E_1 \cup E_2$ , it bounds a properly embedded disk in the 3-ball. If  $a \cup b$  is a meridian of  $\partial N(K_1)$ , a punctured  $S^2 \times S^1$  is embedded in  $S^3$ , a contradiction. If  $a \cup b$  is a longitude of  $\partial N(K_1)$ , then the regular neighborhood of  $N(K_1) \cup D$  is a 3-ball containing  $K_1$  only. This implies  $\overline{S^3 - N(L)}$  is reducible, so  $L$  is a non-trivial split link, hence cannot have an unknotting tunnel, a contradiction. Since  $a$  is an essential arc in  $T_{1,K_1}$ ,  $a \cup b$  winds around  $\partial N(K_1)$  longitudinally more than once. Then a punctured lens

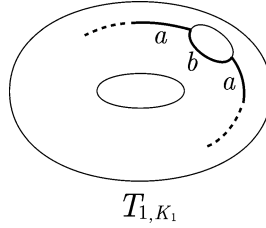


Fig. 7.  $a \cup b$  bounds a disk.

space is embedded in  $S^3$ , a contradiction. Similarly  $\partial D \cap T_{1,K_2}$  should have intersection with  $E_1 \cup E_2$ .  $\square$

Consider an outermost arc  $a$  of  $D \cap (E_1 \cup E_2)$  in  $D$ . Let  $\Delta$  be the outermost disk in  $D$  corresponding to  $a$ . We may assume  $a \subset D \cap E_1$ . Let  $\partial a = \{p, q\}$ . By Lemma 4.2 there are 4 cases according to the labels of the arcs of  $\partial D$  containing  $p$  and  $q$ . For the first three cases, we will show that  $|D \cap (E_1 \cup E_2)|$  can be reduced by removing the arc  $a$ . In the remaining 4th case we show that there exists a stabilizing disk. Notice that  $\partial D \cap (T_{1,K_1} \cup T_{1,K_2}) \neq \emptyset$ . By Lemma 4.2, we have  $|D \cap (E_1 \cup E_2)| > 0$  which guarantees that the case 4 always occurs.

*Case 1.*  $p$  and  $q$  are in one arc labelled  $t_1$ , and the two numbers labelled to  $p$  and  $q$  are  $i$  and  $i + 1$ , respectively (Fig. 8).

Pushing  $E_1$  along  $\Delta$  removes  $a$  and so we can reduce  $|D \cap (E_1 \cup E_2)|$ .

*Case 2.*  $p$  and  $q$  are in one arc labelled  $K_i$  ( $i = 1$  or  $2$ ) (Fig. 9).

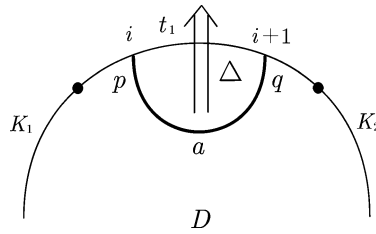


Fig. 8.  $p$  and  $q$  are in one arc labelled  $t_1$ .

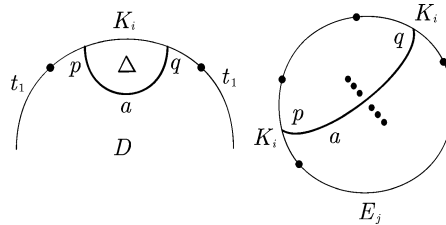
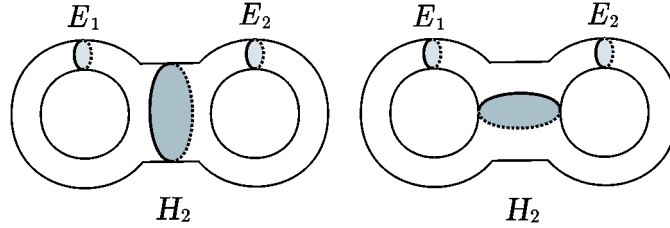


Fig. 9.  $p$  and  $q$  are in one arc labelled  $K_i$  ( $i = 1$  or  $2$ ).

Fig. 10. Two essential disks in  $H_2$  not isotopic to  $E_i$  ( $i = 1, 2$ ).

By cutting and pasting  $E_1$  along  $\Delta$ , we can get two disks  $E'_1$  and  $E''_1$ . By pushing slightly, we can make  $E'_1$  and  $E''_1$  disjoint from  $E_1$  and  $E_2$ .

If  $E'_1$  (respectively  $E''_1$ ) is isotopic to  $E_1$ , we can reduce  $|D \cap (E_1 \cup E_2)|$  by replacing  $E_1$  with  $E'_1$  (respectively  $E''_1$ ). This occurs when  $E''_1$  (respectively  $E'_1$ ) is  $\partial$ -parallel. Suppose that  $E'_1$  and  $E''_1$  are not  $\partial$ -parallel and not isotopic to  $E_1$ . If  $E'_1$  (respectively  $E''_1$ ) is isotopic to  $E_2$ , then  $E''_1 \cup E_2$  (respectively  $E'_1 \cup E_2$ ) cuts  $H_2$  into a 3-ball. Therefore  $|D \cap (E_1 \cup E_2)|$  can be reduced by replacing  $E_1$  with  $E''_1$  (respectively  $E'_1$ ). Now we consider the case that none of  $E'_1$  and  $E''_1$  is isotopic to  $E_i$  ( $i = 1, 2$ ). There are two types of essential disks in  $H_2$  which are not isotopic to  $E_1$  or  $E_2$ —separating and non-separating. See Fig. 10.

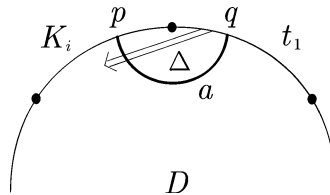
Since  $E'_1$  and  $E''_1$  can be chosen so that  $\partial E'_1$  and  $\partial E''_1$  are boundary components of  $N(E_1 \cup \Delta) \cap \partial H_2$  which is a pair of pants whose third boundary component is isotopic to  $\partial E_1$ , at least one of  $E'_1$  and  $E''_1$ , say  $E'_1$ , is non-separating. Then we can reduce  $|D \cap (E_1 \cup E_2)|$  by replacing  $E_1$  with  $E'_1$ .

And in any case, we can see that the replaced  $E_1$  and  $E_2$  satisfy  $\partial T_{1,K_1} \cap E_i = \emptyset$  ( $i = 1, 2$ ) and  $\partial T_{1,K_2} \cap E_i = \emptyset$  ( $i = 1, 2$ ).

*Case 3.*  $p$  is in an arc labelled  $K_i$  ( $i = 1$  or  $2$ ) and  $q$  is in an adjacent arc labelled  $t_1$  (Fig. 11).

Note that the number labelled to  $q$  is 1 if  $i = 1$  and  $n$  if  $i = 2$ . Suppose the number labelled to  $q$  is 1. Then the meridian disk of  $t_1$  with label 1 cuts the tubular neighborhood of  $t_1$  in  $N(L \cup t_1)$  into two parts. Slide the neighborhood of one of the parts adjacent to  $\Delta$  along and then off  $\Delta$ . Then  $|D \cap (E_1 \cup E_2)|$  is reduced. It is similar in the case that the number labelled to  $q$  is  $n$ .

*Case 4.*  $p$  is in an arc labelled  $K_1$  and  $q$  is in an arc labelled  $K_2$ , where there is one arc labelled  $t_1$  between them (Fig. 12).

Fig. 11.  $p$  is in an arc labelled  $K_i$  ( $i = 1$  or  $2$ ) and  $q$  is in an adjacent arc labelled  $t_1$ .



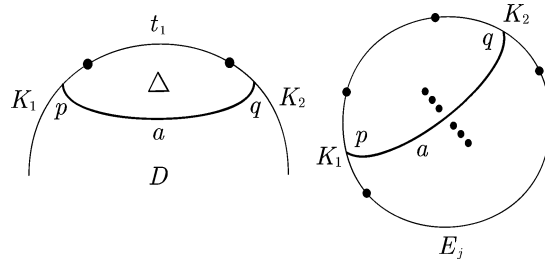


Fig. 12.  $p$  is in an arc labelled  $K_1$  and  $q$  is in an arc labelled  $K_2$ , where there is one arc labelled  $t_1$  between them.

In this case  $t_1$  does not intersect  $E_1 \cup E_2$ . By cutting and pasting  $E_1 \cup E_2$  along  $\Delta$ , we get a stabilizing disk of  $t_1$  for the genus two Heegaard surface  $\partial N(L \cup t_2)$ . So  $\overline{S^3 - N(L \cup t_1 \cup t_2)}$  is a genus three handlebody.

This completes the proof of Theorem 3.1.

## 5. 2-bridge links

Let  $S_i = s_i \cup s'_i$  be a trivial 2-string tangle in a 3-ball  $B_i$  ( $i = 1, 2$ ). Gluing  $B_1$  and  $B_2$  along their boundary spheres so that  $s_1 \cup s_2$  and  $s'_1 \cup s'_2$  are simple closed curves, we obtain a 2-bridge link  $L = S_1 \cup S_2$ . We assume that  $L$  is non-trivial. By [6], there are two types of unknotting tunnels for 2-bridge links. If two unknotting tunnels of  $L$  are parallel, we can easily find a stabilizing disk. So assume that the two tunnels are in standard positions  $t_1$  and  $t_2$  as in Fig. 13.

Note that  $\overline{B_2 - N(S_2)}$  is a genus two handlebody, and  $\overline{B_1 - N(S_1 \cup t_1)}$  is homeomorphic to  $(B_1 - N(S_1 \cup t_1) \cap \partial B_1) \times I$ . Let  $f : \partial B_1 \rightarrow \partial B_2$  be the gluing homeomorphism.

Let  $D_1$  be the meridian disk as in Fig. 13 and  $a = \partial D_1 \cap \partial B_2$ . Let  $b = f(a)$  and  $D_2$  be a disk in  $\overline{B_1 - N(S_1 \cup t_1)}$  corresponding to  $b \times I \subset (B_1 - N(S_1 \cup t_1) \cap \partial B_1) \times I$ . Then  $D = D_1 \cup_f D_2$  is a meridian disk of genus two handlebody  $\overline{S^3 - N(S_1 \cup S_2 \cup t_1)}$  that does not intersect  $t_2$ . So this is an example that satisfies the hypothesis of Theorem 3.1 and  $t_1$  and  $t_2$  give a common stabilization.

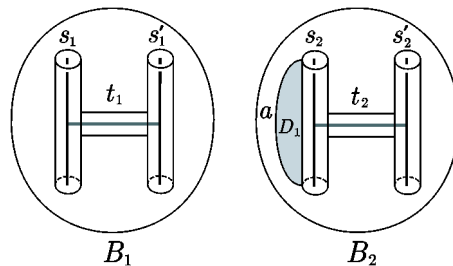


Fig. 13. Unknotting tunnels of a 2-bridge link.

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